Gravity-Induced Angular Motion of a Spinning Missile

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The usual analysis of the steady-state angular motion of a dynamically stable spinning missile assumes a quasi-steady-state calculation of a gravity-induced trim angle. A condition for the validity of this quasi-steady-state assumption is derived. When this condition is not satisfied, the gravity-induced angular motion must be described differently for three distinct portions of the trajectory: the upleg, near apogee, and the downleg. The accuracy of this description is checked by comparison with numerical integratons. Finally the influence of cubic static and Magnus moments on the motion is determined and a revised point mass trajectory model is constructed.

Nomenclature

$C_D, C_{L\alpha}$ = drag and lift coefficients
$C_{M\alpha}$ = static moment coefficient
$C_{M\dot{\alpha}}, C_{M_q}$ = damping moment coefficients
$C_{Mp\alpha}$ = Magnus moment coefficient
G = $Pg_{\hat{z}}lV^{-2}$
$g_{\hat{z}}$ = component of gravity along the $\hat{z}$ axis
g = gravity
H = $(\rho Sl/2m)[C_{L\alpha} - C_D - k_t^{-2}(C_{M_q} + C_{M\dot{\alpha}})] - glV^{-2}\sin\theta_T$
I_x, I_y = axial, transverse moments of inertia
K_j = amplitude of j-mode ($j = 1, 2$)
K_G, K_{2G} = defined in Eq. (19) and Eqs. (25) and (28), respectively
k_a = $(I_x/ml^2)^{1/2}$, axial radius of gyration
k_t = $(I_y/ml^2)^{1/2}$, transverse radius of gyration
l = reference length
M = $(\rho Sl^3/2I_y)C_{M\alpha}$
M_0, M_2 = cubic static moment coefficients in Eq. (30)
m = mass
P = I_xpl/I_yV, gyroscopic spin
p = roll rate
S = reference area
s = dimensionless arclength $\int_{t_0}^{t}(V/l)dt$
s_g = $P^2/4M$, stability factor
T = $(\rho Sl/2m)[C_{L\alpha} + k_a^{-2}C_{Mp\alpha}]$
T_0, T_2 = cubic Magnus moment coefficients in Eq. (30)
t = time
V = magnitude of velocity
$\hat{v}, \hat{w}$ = $\hat{y}$, $\hat{z}$ components of the velocity vector
x_e, y_e, z_e = Earth-fixed Cartesian coordinates
$\hat{x}, \hat{y}, \hat{z}$ = fixed-plane Cartesian coordinates
$\hat{\alpha}, \hat{\beta}$ = angles of attack and sideslip
δ_g = G/M
δ = $|\hat{\xi}|$, sine of total angle of attack
θ_T = angle between trajectory and its projection on the horizontal plane
λ_j = damping rate of the j-mode amplitude ($j = 1, 2$) K_j'/K_j
ξ = $(\hat{v} + i\hat{w})/V$
$\hat{\xi}_g$ = $-G/M$, steady-state, gravity-induced trim angle
$\hat{\xi}_G$ = defined in Eq. (16)
ρ = air density
ϕ_j = j-mode phase angle ($j = 1, 2$) $\phi_{j0} + \phi_j's$

Superscripts

$(\ )^*$ = $(\rho Sl/2m)(\ )$
$(\ )'$ = $d(\ )/ds$
$(\hat{\ })$ = component in fixed-plane coordinate system
$(\dot{\ })$ = $d(\ )/dt$

Subscripts

a = evaluated at apogee
U, D = upleg, downleg

Presented as Paper 70-968 at the AIAA Guidance Control and Flight Mechanics Conference, Santa Barbara, Calif., August 17–19, 1970; submitted September 30, 1970; revision received April 26, 1971.

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Introduction

THE linear angular motion of missiles can usually be written as a sum of responses to various forcing functions and a solution involving the initial conditions. For a dynamically stable missile the effect of initial conditions quickly decays and the angular motion is controlled by the forcing functions, i.e., moments which do not depend on the missile's angles of attack or sideslip or their derivative. For a slowly spinning missile the most important such forcing function is a constant pitch or yaw moment fixed on the missile and caused by either an intentional control surface deflection or an unintentional configurational symmetry. The response to such a moment can take on large values when the pitch rate is near the roll rate and as a result it has been studied by a number of authors.[1–3]

For a symmetric missile with a high spin rate, the forcing function has a magnitude which is proportional to the product of the spin-to-velocity ratio and the trajectory curvature, and has an axis of rotation which is perpendicular to the plane of the trajectory. For a constant amplitude moment and a linear static moment the response is a constant angle of sideslip. This trim sideslip angle causes the nose of a spinning shell to always point to the right and, thereby, produces a right deflection of the trajectory which is called drift.[4–5]

Since both the spin-to-velocity ratio and the trajectory curvature increases to a maximum at apogee, a maximum gravity-induced trim angle is predicted at apogee. This prediction assumes that a quasi-steady-state calculation is appropriate and that the aerodynamic moments are linear. If either of these conditions are not satisfied, a complete six-degree-of-freedom numerical integration is usually required. This paper presents a new simple approximation for this gravity-induced angular motion which is valid for rapidly changing conditions near apogee. The effect of a nonlinear moment is incorporated by use of the quasilinear assumption which has been quite successful for the analysis of the transient motion.[6–8] Finally this approximation is used to obtain a revised version of a modified point mass trajectory.

Equations of Motion

We will make use of two Cartesian axis systems. The first is an Earth-fixed system with the x_e-axis taken as the intersection of the horizontal plane with the plane of the trajectory, the z_e-axis aligned along the gravity vector and the y_e-axis specified by the right-hand rule. The second axis system has the $\hat{x}$-axis along the missile-axis of symmetry, the $\hat{z}$-axis in the plane of the trajectory pointing downward and the $\hat{y}$-axis determined by the right-hand rule. For this fixed plane axis system we make use of the complex angle of attack, $\hat{\xi}$, which is defined by the equation

$$\hat{\xi} \equiv (\hat{v} + i\hat{w})/V = \sin\hat{\beta} + i \cos\hat{\beta} \sin\hat{\alpha} \qquad (1)$$

$$\hat{\xi}_G = \hat{\xi}_g - \int_0^s \frac{\{(\lambda_2 + i\phi_2') \exp[(\lambda_1 + i\phi_1')(s - \hat{s})] - (\lambda_1 + i\phi_1') \exp[(\lambda_2 + i\phi_2')(s - \hat{s})]\}G'(\hat{s})d\hat{s}}{(M + iPT)[\lambda_1 - \lambda_2 + i(\phi_1' - \phi_2')]} \qquad (16)$$

where $\hat{v}$, $\hat{w}$ are $\hat{y}$ and $\hat{z}$ components of the velocity vector and $\hat{\alpha}$, $\hat{\beta}$ are the angles of attack and sideslip. The magnitude of $\hat{\xi}$ is the sine of the angle between the missile's axis and the velocity vector and its orientation determines the orientation of the plane of this angle with respect to the horizontal. For a linear aerodynamics and small geometric angles $\hat{\xi}$ must satisfy the equation[7]

$$\hat{\xi}'' + (H - iP)\hat{\xi}' - (M + iPT)\hat{\xi} = G \qquad (2)$$

where the coefficients are defined in the nomenclature.

The plane trajectory of a particle acted on by gravity and drag can be described by the equations

$$m\ddot{x}_e = -\tfrac{1}{2}\rho V^2 S C_D \dot{x}_e/V \qquad (3)$$

$$m\ddot{z}_e = mg - \tfrac{1}{2}\rho V^2 S C_D \dot{z}_e/V \qquad (4)$$

Introducing θ_T, the inclination of the trajectory with respect to the horizontal these equations can be written in the form

$$V'/V = -C_D^* - glV^{-2} \sin\theta_T \qquad (5)$$

$$\theta_T' = -glV^{-2} \cos\theta_T \qquad (6)$$

where $C_D^* = (\rho Sl/2m)C_D$. Equations (5) and (6) can be integrated for constant C_D^* to give the velocity as a function of trajectory angle.

$$V = V_a \sec\theta_T\{1 - (C_D^* V_a^2/gl)[\tan\theta_T \sec\theta_T] + \ln \tan[(\theta_T/2) + \pi/4]\}^{1/2} \qquad (7)$$

where V_a is velocity at apogee.

The gravity terms in Eq. (2) can now be approximated by θ_T if we assume a small angle of attack

$$g_{\hat{z}} \doteq g \cos\theta_T \qquad (8)$$

$$\therefore G = -P\theta_T' \qquad (9)$$

The solution to Eq. (2) for slowly varying coefficients is[6]

$$\hat{\xi} = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} + \hat{\xi}_g \qquad (10)$$

where

$$\phi_j' = \tfrac{1}{2}[P \pm (P^2 - 4M)^{1/2}] \qquad (11)$$

$$K_j'/K_j \equiv \lambda_j = -[H\phi_j' - PT + \phi_j'']/(2\phi_j' - P) \qquad (12)$$

$$\hat{\xi}_g = -G/(M + iPT) \doteq -G/M \qquad (13)$$

The expression for the gravity-induced trim angle, $\hat{\xi}_g$, is based on the the quasi-steady-state assumption that G, and M vary slowly during a cycle of the transient epicyclic motion given by the first two terms of Eq. (10) (See Fig. 1).

Gravity-Induced Trim without Damping

Near apogee G varies rapidly as can be seen from its derivative for constant spin

$$G' = (-V'/V + \theta_T''/\theta_T')G = (3C_D^* + 4glV^{-2} \sin\theta_T)PglV^{-2} \cos\theta_T \qquad (14)$$

The relative variations of G and G' are indicated in Fig. 2. The maximum value of G occurs after apogee due to the action of drag. To obtain the angular response to G we make use of the method of variation of parameters for the simple case of constant λ_j's and ϕ_j''s and integrate the result by parts:

$$\hat{\xi} = K_1 e^{i\phi_1} + K_2 e^{i\phi_2} + \hat{\xi}_G \qquad (15)$$

For a gyroscopically stabilized missile the fast rate ϕ_1' is usually much greater than the slow rate ϕ_2', especially near apogee where G' is large. Integrals of complex exponentials are inversely proportional to their frequencies and, thus, we can easily neglect the first term in the integral in comparison with the second term and for simplicity we will neglect λ_j in in comparison with ϕ_j' in the multiplying coefficients but not in the exponential coefficients

$$\hat{\xi}_G = -\delta_g + (1/M) \int_0^s \{\exp[(\lambda_2 + i\phi_2')(s - \hat{s})]\}G'(\hat{s})d\hat{s} \qquad (17)$$

where $\delta_g = G/M$. Although Eq. (17) is a simple relation for the gravity-induced trim angle, it is gravely limited by the restriction to a constant frequency. For most projectiles the apogee value of the gyroscopic stability factor usually exceeds ten when G' is large enough to affect Eq. (17) and a quite simple expression for ϕ_2' can be written from Eq. (11)

$$\phi_2' = (P/2)[1 - (1 - 1/s_g)^{1/2}] = (M/P)[1 + 1/4s_g + \ldots] \doteq M/P \qquad (18)$$

where $s_g = P^2/4M$. Since P is proportional to the spin-to-velocity ratio and the spin normally decays quite slowly due to viscous damping, P can grow quite rapidly on the upleg and, therefore, the assumption of constant ϕ_2' is not satisfied. A reasonably good first approximation for the effect of varying frequency on the derivation of Eq. (17) is to replace $(\phi_2')(s)$ by

$$\phi_2 = \int_{s_a}^s \phi_2' ds$$

$$\therefore \hat{\xi}_G = -\delta_g + \delta_{ga} K_G \exp[\lambda_2(s - s_a) + i\phi_2(s)] \qquad (19a)$$

where

$$K_G = G_a^{-1} \int_0^s \exp\{-[\lambda_2(\hat{s} - s_a) + i\phi_2(\hat{s})]\}G'(\hat{s})d\hat{s} \qquad (19b)$$

Eq. (19) clearly reduces to the quasi-steady state relation when G' can be neglected. Indeed K_G can be neglected when the

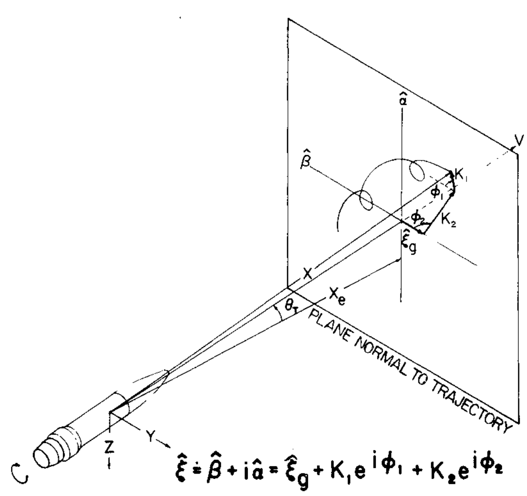

Fig. 1 Angular motion of spinning projectile.

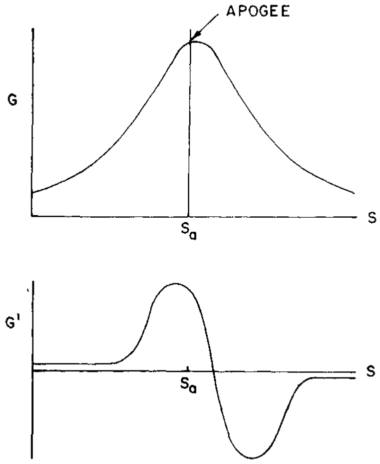

Fig. 2 Variation of G and G' over the flight path.

width of the humps of G' is large with respect to the wavelength of ϕ_2.

The basic properties of Eq. (19) can be determined if we consider the very simple case of zero drag, constant spin rate ($\dot{p} = 0$) and no aerodynamic damping. For zero drag Eq. (7) reduces to

$$V = V_a \sec\theta_T \tag{20}$$

and

$$G' = 4(glV_a^{-2})G_a \cos^6\theta_T \sin\theta_T \tag{21}$$

Equation (18), then, gives an approximation for ϕ_2'

$$\phi_2' \doteq (M/P_a)\sec\theta_T = (glV_a^{-2})\delta_{ga}^{-1}\sec\theta_T \tag{22}$$

The integral for K_G now assumes a very simple form for no damping

$$K_G(\phi_2,\delta_{ga}) = \delta_{ga}\int_{\phi_{20}}^{\phi_2}\{\exp(-i\hat{\phi}_2)\}([f(\theta_T)d\hat{\phi}_2] \tag{23}$$

where $f(\theta_T) = 4\cos^7\sin\theta_T$. Finally a relationship between θ_T and ϕ_2 can be obtained from Eqs. (6) and (22)

$$\phi_2 = -\delta_{ga}^{-1}\tan\theta_T[3 + \tan^2\theta_T]/3 \tag{24}$$

$f(\theta_T)$ is plotted vs ϕ_2 for various values of δ_{ga} in Fig. 3.

A brief examination of K_G shows that it is essentially constant for ϕ_2 outside the interval $(-2\pi,2\pi)$. On the upleg portion of the trajectory $(\phi_2 < -2\pi)K_G$ is zero while on the downleg portion $(\phi_2 > 2\pi)$ it has a zero real part. This situation can be summarized by the following equation

$$\hat{\xi}_G = -\delta_g + \delta_{ga}K_G(\delta_{ga},\phi_2)\exp i\phi_2 \tag{25a}$$

where

$$K_G = 0 \quad \phi_2 < -2\pi \quad \text{upleg} \tag{25b}$$

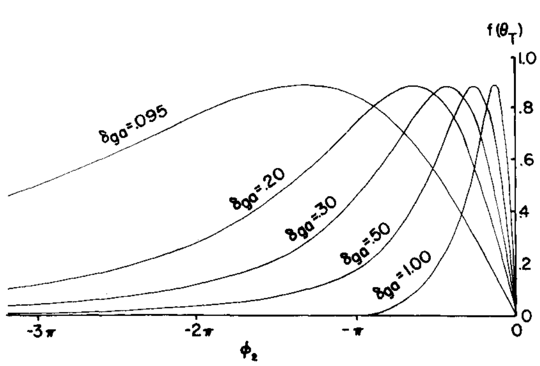

Fig. 3 Variation of $f(\theta_T)$ as a function of the slow mode phase shift from apogee for several values of δ_{ga}.

$$K_G = \delta_{ga}\int_{-2\pi}^{\phi_2}\{\exp(-i\hat{\phi}_2)\}[f(\theta_T)d\hat{\phi}_2] \quad -2\pi < \phi_2 < 2\pi \quad \text{near apogee} \tag{25c}$$

$$K_G = K_G(\delta_{ga},2\pi) \quad 2\pi < \phi_2 \quad \text{downleg} \\ = iK_{2G}(\delta_{ga}) \tag{25d}$$

where

$$K_{2G} = -\delta_{ga}\int_{-2\pi}^{2\pi} f(\theta_T)\sin\hat{\phi}_2 d\hat{\phi}_2 \tag{25e}$$

$K_{2G}(\delta_{ga})$ is given as a function of δ_{ga} in Fig. 4. An important feature of this curve is that K_{2G} is quite small for $\delta_{ga} < 0.15$, and thus we would expect the quasi-steady-state results to be good when the predicted apogee steady-state angle is less than 8°. When the steady-state prediction exceeds this value, Eq. (25) or the more accurate Eq. (19) should be used. K_{2G} can be identified as a fraction of δ_{ga} which appears impulsively at the apogee in the slow mode when the gravity forcing function is not varying slowly in a period of the slow mode.

In Fig. 5 the combined pitching and yawing motion for a missile with a large maximum gravity-induced trim ($\hat{\beta} = 20°$) is shown. The parameters used in this exact integration are given in Table 1. From Fig. 4 we see that K_{2G} is .76 and, therefore, an initial value for K_2 of 15° is indicated.

This motion starts out as an epicyclic motion induced by an initial angular velocity. During the first three seconds the center of this epicycle moves to the right; then it moves up as well as continuing its right motion until eight seconds. There then appears a rough reversal of this process until eleven seconds is reached. After this point a new epicyclic motion is established with a much larger slow mode motion with an amplitude of 12°. This qualitative behavior is precisely that predicted by Eq. (25) with the near apogee motion occurring between three and eleven seconds. The terminal slow mode amplitude of 12° is quite consistent with an apogee value of 15° when the influence of aerodynamic damping is computed.

Gravity-Induced Trim with Damping

The effect of constant damping is included in Eq. (19). Near apogee ϕ_2' is much smaller than P and from Eq. (12) we see that a good approximation for λ_2 is $-T$, which can be constant for near apogee flight. For this case Eq. (25) takes on the revised form

$$\hat{\xi}_G = -\delta_g + \delta_{ga}K_G(\delta_{ga},\phi_2)\exp[-T(s - s_a) + i\phi_2] \tag{26a}$$

where

$$K_G = \delta_{ga}\int_{-2\pi}^{\phi_2}\exp[T(\hat{s} - s_a) - i\hat{\phi}_2][f(\theta_T)d\hat{\phi}_2] \quad -2\pi < \phi_2 < 2\pi \tag{26b}$$

On the downleg portion ($s > s_D$) of the flight ϕ_2' grows and H and T vary as the Mach number increases. During this portion of flight G' is quite small and the integral in Eq. (19) becomes constant. We, therefore, assume the major effect of G' is to specify an initial value of K_2 and use Eq. (12) to predict the influence of varying λ_2

$$\hat{\xi}_G = -\delta_g + K_2\exp[i(\phi_2 + \phi_{2G})] \tag{27}$$

$$K_2 = \delta_{ga}K_{2G}\exp\left(\int_{s_a}^{s}\lambda_2 d\hat{s}\right) \tag{28a}$$

Table 1 Parameters for exact integration

V_0	= 255 fps	δ_{ga}	= 0.335 ($\hat{\beta}$ = 20°)
$\theta_T(0)$	= 60°	δ_{g0}	= 0.021 ($\hat{\beta}_g$ = 1.2°)
M	= 1.5×10^{-4}	ϕ_{1a}'	= 0.071
T	= 1.5×10^{-4}	ϕ_{2a}'	= 0.002
H	= 2.9×10^{-4}	$\hat{\xi}'(0)$	= $-i$ (0.5) rad/sec
P_0	= 0.036	$\hat{\xi}(0)$	= 0
C_D*	= 0		

where

$$K_{2G}\exp(i\phi_{2G}) = G_a^{-1}\int_{sU}^{sD}\exp[T(\hat{s} - s_a) - i\phi_2(\hat{s})][G'd\hat{s}] \quad (28b)$$

For zero drag a simple expression for K_{2G} can be obtained:

$$K_{2G} = \left|\delta_{ga}\int_{-2\pi}^{2\pi}\exp[T(\hat{s} - s_a) - i\phi_2][f(\theta_T)d\phi_2]\right| \quad (29)$$

If $|T/\phi_2'| < 0.1$, actual numerical calculations show that K_{2G} is within .02 of its value for $T = 0$ and, hence, Fig. 4 can be used to obtain K_{2G} as a function of δ_{ga}.

Nonlinear Analysis

The usual quasi-linear analysis[6–8] has been applied primarily to the angular motions of symmetric missiles with no moment forcing functions. This analysis has recently been extended to include the forcing function associated with slight configurational asymmetries.[3] The latter treatment can be easily extended to include gravity-induced angular motion away from apogee.[6–9] In this section we will outline the appropriate analysis and give the results for a cubic static and Magnus moments. For this case Eq. (2) becomes

$$\hat{\xi}'' + (H - iP)\hat{\xi}' - [M_0 + M_2\delta^2 + iP(T_0 + T_2\delta^2)]\hat{\xi} = G \quad (30)$$

where $\delta^2 = |\hat{\xi}|^2$. A solution of the form of Eq. (15) is assumed and substituted in Eq. (30). The resulting equation is divided by $K_2 \exp i\phi_2$ and averaged over a distance which is large with respect to the wavelength of the slow rate to yield quasi-linear values of λ_2 and ϕ_2';

$$\lambda_2 = -[H\phi_2' - P(T_0 + T_2\delta_{e2}^2) + \phi_2'']/(2\phi_2' - P) \quad (31)$$

$$\phi_2' = \tfrac{1}{2}\{P - [P^2 - 4(M_0 + M_2\delta_{e2}^2)]^{1/2}\} \quad (32)$$

where $\delta_{e2}^2 = K_2^2 + 2\delta_g^2$. If the resulting equation is divided by $K_1 \exp i\phi_1$, similar relations for the high frequency motion follow. Finally the equation can be averaged as it is to yield a quasi-linear relation for the gravity-induced trim

$$-[M_0 + M_2\delta_{e3}^2 + iP(T_0 + T_2\delta_{e3}^2)]\hat{\xi}_g = G \quad (33)$$

where $\delta_{e3}^2 = \delta_g^2 + 2K_2^2$. Since the imaginary part of the coefficient of $\hat{\xi}_g$ is usually less than a quarter of the real part, it only affects the orientation of ξ_g much more than it affects its magnitude, δ_g. A simple equation for δ_g can be written

$$\delta_g = G/[M_0 + M_2(\delta_g^2 + 2K_2^2)] \quad (34)$$

On the downleg Eqs. (31–32) can be used in Eq. (28) to calculate the magnitude of the slow mode motion which has been initiated by G' at the apogee. The orientation of the slow mode motion can be obtained by integrating Eq. (32).

The nonlinear analysis for near apogee motion is much more difficult since δ_g varies rapidly during a cycle of ϕ_2. An estimate of the effect of a cubic static moment can be made for small values of K_2. The steady-state formulas for δ_g then reduce to a cubic equation

$$\delta_g = G/(M_0 + M_2\delta_g^2) \quad (35)$$

The slow frequency, which assumes the form

$$\phi_2' = (M_0 + 2M_2\delta_g^2)\sec\theta_T/P_a \quad (36)$$

varies in response to the nonlinearity as δ_g grows from zero to δ_{ga}. If the nonlinearity term in Eq. (36) is replaced by its average, this equation can be reduced to Eq. (22) of the linear theory

$$\phi_2' = (M_0 + M_2\delta_{ga}^2)\sec\theta_T/P_a = glV_a^{-2}\delta_{ga}^{-1}\sec\theta_T \quad (37)$$

Thus, an approximation for K_G and K_{2G} when the static

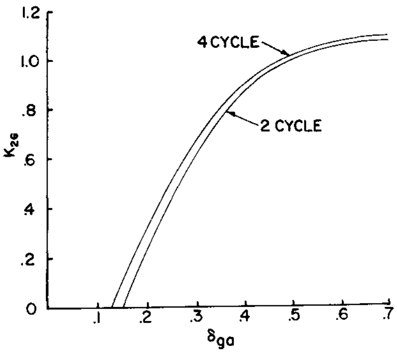

Fig. 4 K_{2G} vs δ_{ga} for $\lambda_2 = 0$. For $|T_2/\phi_2'| < 0.1$ the change in K_{2G} is less than 0.02.

moment is a cubic function can be made by using Eq. (25) with δ_{ga} given by Eq. (35) evaluated at the apogee.

A Revised Point-Mass Trajectory

For many years ordnance firing tables were computed by use of the point mass Eqs. (3–4). These equations completely neglect induced drag due to $\hat{\xi}_G$ as well as lateral drift caused by this angle. The induced drag is accounted for by adjusting C_D by a form factor which is a function of θ_T and is determined by full-range firing. Drift is measured by full-range firings and numerically interpolated for firing table use.

Recently a modified point mass analysis has been developed[10] which includes the effects of the steady-state gravity-induced trim $\hat{\xi}_g = -\delta_g$. This trim angle modified the drag coefficient as well as causing a lateral deflection

$$C_D = C_{D_0} + C_{D\delta^2}\delta_g^2 \quad (38)$$

$$m\ddot{y}_e = \tfrac{1}{2}\rho V^2 SC_{L\alpha}\delta_g \quad (39)$$

This modified point mass trajectory has the advantage of retaining the major trajectory contribution of the angular motion without requiring the use of the very small integration interval associated with an exact integration of Eq. (2). It is valid for a dynamically stable missile and slowly varying G.

The theory of this report can be used to construct an improved version of the modified point mass trajectory which could be called a revised point mass trajectory. Since the motion near and after apogee involves the slow frequency, an integration interval small with respect to the slow mode's period is needed. The integration interval required for Eq. (2) is small with respect to the fast mode's period and

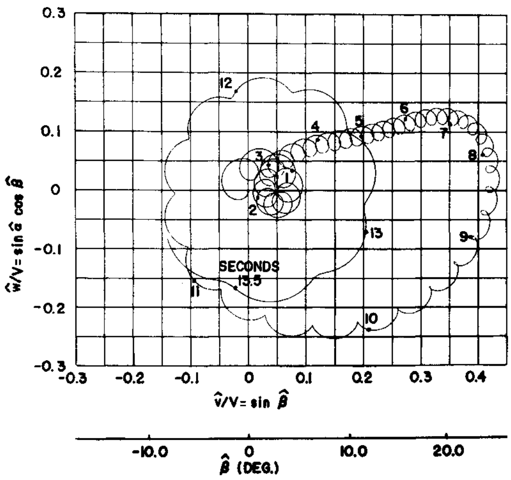

Fig. 5 Pitching and yawing motion of 4.2 mortar shell.

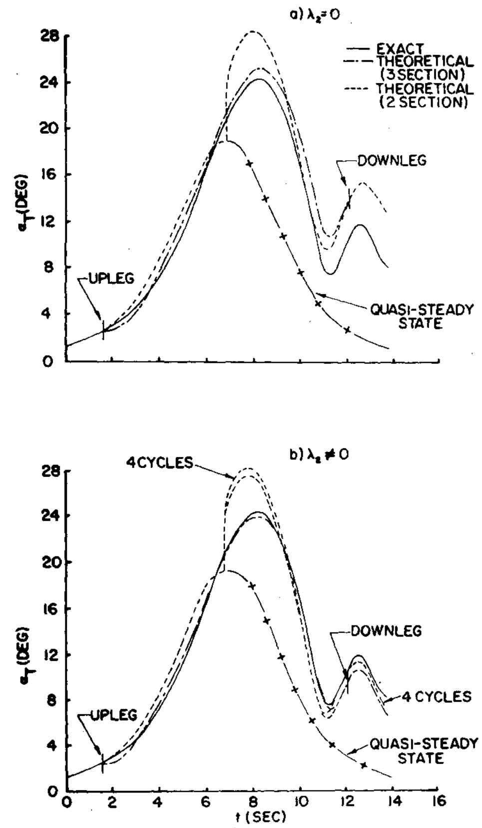

Fig. 6 Comparison of exact values of total angle of attack with various theoretical approximations.

is, therefore, much smaller than that required for the revised point mass trajectory. Thus the revised point mass trajectory requires much less computer time than the exact six-degree-of-freedom trajectory.

It has been shown[7] that only the average of δ^2 need be considered for the drag force. If we can neglect the effect of K_G on drag near apogee where the total drag force is small, only the drag on the downleg need be revised.

$$C_D = C_{D_0} + C_{D\delta^2}(\delta_g^2 + K_2^2) \tag{40}$$

The limitation imposed on the integration interval by the slow mode motion can now be eliminated if the lateral deflection due to K_G can be neglected. The lateral deflection due to K_G is caused by an angle which is constantly changing direction. The average value of this deflection angle can be estimated by calculating the jump angle for a projectile performing coning motion of magnitude K_2 and frequency ϕ_{2a}'

$$\text{Jump angle} = C_{L\alpha}^* K_2/\phi_{2a}' \tag{41}$$

This jump angle is a right deflection angle of the impact point with respect to the apogee. The deflection angle with respect to the gun is one-half this angle and usually quite small. The refined point mass trajectory, then, requires the same integration interval as the modified point mass trajectory. If the effect of K_G on drag near apogee is required, a smaller integration interval will be required for this small portion of the trajectory.

Comparison with Exact Theory Evaluations

In order to make a direct comparison with exact calculations initial conditions of $\xi_0 = -\delta_g$, $\xi_0' = -\delta_{g0}'$ were used with the other parameters of Table 1 to give an angular motion without a transient epicycle. The total angle-of-attack variation with time for these conditions is shown in Fig. 6a and is compared with the quasi-steady-state δ_g and the angular motion given by Eq. (25). We see that the prediction of Eq. (25) is much better than that of the quasi-steady-state theory but it does overestimate α_t by 35%.

The calculations based on Eq. (25) can be considerably simplified if the near apogee motion is approximated by a discontinuous jump at apogee from the $-\delta_g$ motion before apogee to the $-\delta_g + K_2 \exp(i\phi_2)$ motion after apogee. This calculation which considers only two sections of the trajectory is also given in Fig. 6a and with exception of a region very close to apogee it is seen to be a good approximation to the three-section theory.

Finally the effect of damping is calculated through Eqs. (26–28). The two- and three-section calculations were repeated for nonzero damping and are plotted in Fig. 6b. Here we see that the theory underestimates α_t by about 15% near t = 13 sec. This discrepancy, however, is entirely due to calculating K_{2G} over a two-cycle interval, i.e., one cycle on both sides of apogee. K_{2G} was then calculated over a four-cycle interval (two cycles on both sides of apogee) and the result is plotted as Fig. 4. This shows a difference of about 5%. The two-section calculation is repeated in Fig. 6b using the four-cycle integration value of K_{2G} and we see the agreement with the exact curve to be quite good.

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